

Contextual Terminologies

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Abstract. The paper addresses the issue of contextual representations of ontologies, as it arises in the area of normative system specifications for modeling multi-agent systems. To this aim, the paper proposes a formalization of a notion of contextual terminology, that is to say, a terminology holding only with respect to a specific context. The formalization is obtained by means of a formal semantics framework which enables the expressivity of common description logics to reason within contexts (intra-contextual reasoning), allowing at the same time the possibility to reason also about contexts and their interplay (inter-contextual reasoning). Using this framework, two complex scenarios are discussed in detail and formalized.

1 Introduction

The present research is motivated by problems stemming from the domain of normative system specifications for modeling multi-agent systems ([5, 19, 10]). In [6, 18] contexts have been advocated to play a central role in the specification of complex normative systems and normative systems of high complexity (for instance legal systems, or institutional ones) are viewed not only as regulative systems, but also as systems specifying conceptualizations, or categorizations, of the domain of entities they are supposed to regulate. In [12, 9] we proposed and applied a framework for representing this categorizing feature of normative systems via contextual taxonomic statements of the form “A counts as B in context C” ([16]), where concept descriptions A and B displayed a very simple logical form (essentially boolean compositions of concepts). This work intends to pursue that research line further adding the necessary expressivity (essentially the possibility to deal with attributes or roles, i.e., binary relations besides concepts) to model more complex scenarios: from simple taxonomies to rich description logic terminologies.

The final aim consists in obtaining a framework in which to represent ontologies of different contexts and to reason about them both in isolation, i.e., within the contexts (intra-contextual reasoning), and in interaction, i.e. between contexts (inter-contextual reasoning). For instance, at the intra-contextual level a typical question would be of the form: given a set of subsumption relations holding in context C, is A a subconcept of B in context C? At an inter-contextual level instead, a typical question would be: given that context C is more concrete

than context D, is A a subconcept of B in context C? With such a machinery it would then be possible to represent the ontological aspect of the regulating activity of institutions in a formal way, and the ontologies of different institutions could then be rigorously specified and reasoned about. To do this, we show that the approach proposed in [12] can be naturally applied to richer description logic languages thus providing the necessary expressive power we are interested in. In fact, the framework presented here consists in a contextualized version of the semantics of description logics. The proposal is tested in detail against two different examples.

The exposition is structured according to the following outline. In Section 2 two scenarios are introduced which exemplify in detail the issues addressed here, and some preliminary considerations are drawn. Section 3 is dedicated to the exposition of the framework, and Section 4 to the formalization of the two scenarios introduced in Section 2. Conclusions follow.

2 Preliminaries

2.1 Scenarios

We now depict two scenarios in order to state, in clear terms, the kind of reasoning patterns we are aiming to capture formally. They exemplify quite typical forms of contextual conceptualizations occurring in the normative domain. The first scenario deals with a rule establishing sufficient conditions for a person to be liable of violating the regulation concerning access to public parks in three different municipalities. The second scenario deals with the refinement of a definition of “vehicle” from the abstract context of a general regulation to more concrete contexts of municipal regulations. From a logical point of view, they display description logic forms of reasoning at the level of the so-called taxonomical boxes (TBoxes)¹ (e.g., reasoning with value restriction and existential quantification, role subsumption) which were not yet available in our previous proposal ([12]).

Example 1. (The public park scenario: “liability in parks”) In the regulation governing access to public parks in region R it is stated that vehicles are not allowed within public parks and that: “persons using vehicles within public parks are liable for violating the regulation”. In this regulation no mention is made of (possible) subconcepts of the concept vehicle, e.g., cars, bicycles, which may help in identifying an instance of vehicle, nor it is stated what it actually means to drive a vehicle: does the fact that I am wheeling my bicycle imply that I am driving it? In municipal regulations subordinated to this regional one, and therefore inheriting its global directives, specific subconcepts are instead handled. In municipality M1 and M2 the following rule holds: “persons driving bicycles within parks are liable of violating the regulation”. In M3 instead, it

¹ Taxonomical boxes or *terminologies* are, in the description logic vocabulary, sets of inclusion relations between concepts.

holds that to drive a bicycle does not constitute any violation. On the other hand, in all M1, M2 and M3 it holds that cars are not allowed in public parks. Moreover, in M2 it holds that “persons wheeling bicycles into public parks are not liable for violating the regulation” despite liability arises in case bicycles happen to be driven. In M1 and M3 instead, to wheel a bicycle is considered a way of driving it.

| | DRIVE VEHICLE | DRIVE CAR | DRIVE BICYCLE | WHEEL BICYCLE |
|----|---------------|-------------------------|-------------------------|-------------------------|
| R | liable | <i>not classifiable</i> | <i>not classifiable</i> | <i>not classifiable</i> |
| M1 | liable | liable | liable | liable |
| M2 | liable | liable | liable | not liable |
| M3 | liable | liable | not liable | not liable |

Table 1. Liability in the public park scenario

In this scenario the concept of `vehicle` gets various interpretations. Instances of `car` (w.r.t. the terminologies presupposed by M1, M2 and M3) are always instances of `vehicle`, while instances of `bicycle` are only in some contexts also instances of `vehicle`. What also gets various interpretations is the relation driving: somehow driving in M2 has a different meaning than in M1 and M3. Table 1 displays how sufficient conditions for determining liability come down to be interpreted in three completely different ways by the contexts at issue, although in all contexts it holds that persons driving vehicles are to be considered liable. Note that context R cannot provide any qualification for actions such as driving or wheeling a bicycle simply because its language cannot express those notions.

Example 2. (The public park scenario: “teenagers on skateboards”)
 Consider again a regulation governing access to public parks in region R where it is stated that: “vehicles are not allowed within public parks”. Also in this regulation no mention is made of (possible) subconcepts of the concept vehicle. Nevertheless, a (partial) definition, specifying necessary conditions for something to be a vehicle, is stated: “vehicles are conveyances which transport persons or objects”. In municipal regulations subordinated to this regional one subconcepts are instead introduced. This is done inheriting the definition stated at the R level and refining it either incrementing the number of necessary conditions for something to be considered a vehicle or stating sufficient ones. In municipality M1 the definition of vehicle is refined in the following sense: “self-propelled conveyances which transport persons or objects are vehicles” and “vehicles are self-propelled”. In M2, instead, the definition of vehicle is simply closed without any refinement: “conveyances which transport persons or objects are vehicles”. Besides, in both M1 and M2, it holds that “skateboards are conveyances which are not self-propelled” and “teenagers are persons”. These rules determine a different behavior of M1 and M2 with respect to concepts such as “skateboards

transporting teenagers”. With respect to this concept the following rule holds in M1: “skateboards transporting teenagers are not vehicles”. In M2 instead, it holds that: “skateboards transporting teenagers are vehicles”.

The second scenario displays some other aspects of contextual conceptualizations. The concept of `vehicle` gets again various interpretations and is first specified in its necessary conditions by context R and then completely defined in the two concrete contexts M1 and M2. The abstract regulation states that all vehicles are conveyances transporting persons or objects, leaving thus open the possibility for some of such conveyances not to be vehicles. This is the case of skateboards in M1 since M1 refines the abstract rule establishing more necessary conditions (being self-propelled) for conveyances to be classified as vehicles. Context M2 instead, simply closes the abstract rule through establishing that being a conveyance transporting persons or objects is sufficient for being a vehicle. Because of this, the two contexts M1 and M2 validate terminologies diverging on the conceptualization of the complex concept “skateboards transporting teenagers”.

These two scenarios exemplify interesting nuances typical of complex context-dependent conceptualizations². We will constantly refer back to them in the remainder of the work, and our central aim will be to develop a formal semantics framework able to represent analogous scenarios and to provide thus a rigorous understanding of the forms of reasoning therein involved.

2.2 Contextualizing Terminologies

We want to devise a language and a semantics for talking about contextual terminologies. More in detail, this turns out to devise a formal morphology and a formal semantics meeting the following requirements.

Firstly, it should support reasoning about the validity of TBoxes with respect to contexts giving a semantics to expressions of the type: “the concept `bicycle` is a subconcept of the concept `vehicle` in context M1”. Besides this, the framework should be able to express the fact that concepts may be unclassifiable within specific contexts, that is, that specific subsumptions cannot be said to be valid or not valid: in the context R of the regional regulation, whether a person wheeling a bicycle within a public park is to be considered liable of violating the regulation corresponds to a non evaluable subsumption since the concept at issue is not part of the language of the context R (see Figure 2.1). In some sense, it corresponds to a subsumption which is evaluated with respect to the wrong context. Therefore, we want the framework to be able to express whether a concept gets meaning within a context: “concept `bicycle` is meaningful with respect to context M1”. Completely analogous expressions should be available in order to handle a contextualization of role (or attribute) hierarchies such as: “role `wheel (wheeling)` is a subrole of `drive (driving)` in context M2”

² It is instructive to notice that both scenarios represent instances of a typical form of contextual reasoning called “categorization” ([3]), or “perspective” ([1]), that is, the form of reasoning according to which a same set of entities is conceptualized in many different ways.

and “role wheel is meaningful in context M2”. Secondly, it should provide a representation of context interplay. In particular, we will introduce: a *contextual disjunction* operator and a *contextual focus* operator³.

The first one yields a union of contexts: the contexts “viruses” and “bacteria” can be unified on a language talking about microorganisms generating a more general context like “viral or bacterial microorganisms”. The second one, which plays a central role in our framework, yields the context consisting of some information extracted from the context on which it is focused: the context categorizing “crocodiles”, for instance, can be obtained via focusing the context which categorizes all reptiles on the language talking only about crocodiles and disregarding other reptiles. In other words, the operator prunes the information contained in the context “reptiles” focusing only on what is expressible in the language which talks about crocodiles and abstracting from the rest. Also *maximum* and *minimum* contexts will be introduced: these will represent the most general, and respectively the most specific, contexts on a language⁴. It is important to notice that all operations explicitly refer to a precise language on which the operation should take place. As we will see in the following section our formal language will be tuned to incorporate this feature.

Finally, our language should represent specific relations between contexts. Examples 1 and 2 consider groups of contexts in which all contexts are specializations of an abstract one (R). This suggests the consideration of a generality relation between contexts⁵ expressing that a context is at most as general as another one: the context of the abstract regulation R is somehow more general than the concrete ones M1 and M2⁶.

These intuitions about the semantics of context operators will be clarified and made more rigorous in Section 3.2 where the semantics of the framework will be presented, and in Section 4 where the examples will be formalized deploying all these types of expressions.

3 A Formal Framework

Our proposal consists in mixing the semantics of description logic ([2]) with the idea of modeling contexts as sets of models ([7]), delivering a framework able to represent reasoning about sets of concept subsumptions, i.e., taxonomical boxes (TBoxes), in a contextual setting.

³ In [12, 11] the *focus* operation is called *abstraction*. We decided to modify our terminology in order to avoid confusions with other approaches to notions of abstraction like for instance [8].

⁴ In this paper, we limit the number of context operations to disjunction and focus. More operations are formalized in [12]. It is worth noticing, in passing, that similar operations and special contexts are discussed in [17].

⁵ Literature on context theory often addresses this type of relation between contexts. See for instance [15, 3].

⁶ As the discussion of the formalization of the examples will show (Section 4), there are some more subtleties to be considered since R is not only more general but is also specified on a simpler language.

3.1 Language

The language we are defining can be seen as a meta-language for TBoxes defined on \mathcal{AL} description logic languages, which handle also concept union, full existential quantification (we want to deal with concepts such as “either car or bicycle” and “persons who drive cars”) and role complement (we want to be able to talk about roles such as “not driving”)⁷.

The alphabet of the language \mathcal{L}^{CT} (*language for contextual terminologies*) contains therefore the alphabets of a family of languages $\{\mathcal{L}_i\}_{0 \leq i \leq n}$. This family is built on the alphabet of a given “global” language \mathcal{L} which contains all the terms occurring in the elements of the family. Moreover, we take $\{\mathcal{L}_i\}_{0 \leq i \leq n}$ to be such that, for each non-empty subset of terms of the language \mathcal{L} , there exist a \mathcal{L}_i which is built on that set and belongs to the family. Each \mathcal{L}_i contains two non-empty finite sets \mathbf{A}_i of atomic concepts (A), i.e., monadic predicates, and \mathbf{R}_i of atomic roles (R), i.e., dyadic predicates. These languages contain also concepts and roles constructors. As to concept constructors, each \mathcal{L}_i contains the zeroary operators \perp (bottom concept) and \top (top concept), the unary operator \neg (complement), and the binary operators \sqcap and \sqcup . As to role constructors, each \mathcal{L}_i contains the unary operator \overline{R} (role complement). Finally, the value restriction operator $\forall R.A$ (“the set of elements such that all elements that are in a relation R with them are instances of A ”) applies to role-concept pairs.

Besides, the alphabet of \mathcal{L}^{CT} contains a finite set of context identifiers \mathbf{c} , two families of zeroary operators $\{\perp_i\}_{0 \leq i \leq n}$ (minimum contexts) and $\{\top_i\}_{0 \leq i \leq n}$ (maximum contexts), one family of unary operators $\{fcs_i\}_{0 \leq i \leq n}$ (contextual focus operator), one family of binary operators $\{\gamma_i\}_{0 \leq i \leq n}$ (contexts disjunction operator), one context relation symbol \preceq (context c_1 “is less general than” context c_2), two meaningfulness relation symbols “ $\cdot \downarrow^c \cdot$ ” (concept A is meaningful in context c) and “ $\cdot \downarrow^r \cdot$ ” (role R is meaningful in context c), and finally two contextual subsumption relation symbols “ $\cdot : \cdot \sqsubseteq^c \cdot$ ” (within context c , concept A_1 is a subconcept of concept A_2) and “ $\cdot : \cdot \sqsubseteq^r \cdot$ ” (within context c , role R_1 is a subrole of role R_2) for, respectively, concept and role subsumption⁸. Lastly, the alphabet of \mathcal{L}^{CT} contains also the sentential connectives \sim (negation) and \wedge (conjunction)⁹.

Thus, the set Ξ of context constructs (ξ) is defined through the following BNF:

$$\xi ::= c \mid \perp_i \mid \top_i \mid fcs_i \xi \mid \xi_1 \gamma_i \xi_2.$$

⁷ This type of language is indeed an \mathcal{ALC} conceptual language extended with role complement. See [2].

⁸ We use superscripts here in order to distinguish between meaningfulness of concepts or roles, and subsumptions of concepts or roles. Nevertheless, in what follows, superscripts will be dropped when no confusion arises in order to lighten the notation.

⁹ It might be worth remarking that language \mathcal{L}^{CT} is, then, an expansion of each \mathcal{L}_i language. Notice also that all operators on contexts are indexed with the language on which the operation they denote takes place.

Concept constructs and role constructs are defined in the standard way. The set P of roles descriptions (ρ) is defined through the following BNF:

$$\rho ::= R \mid \bar{\rho}.$$

The set Γ of concept descriptions (γ) is defined through the following BNF:

$$\gamma ::= A \mid \perp \mid \top \mid \neg\gamma \mid \gamma_1 \sqcap \gamma_2 \mid \forall\rho.\gamma.$$

Concept union and existential quantification are defined respectively as:

$$\gamma_1 \sqcup \gamma_2 =_{\text{def}} \neg(\neg\gamma_1 \sqcap \neg\gamma_2) \text{ and } \exists\rho.\gamma =_{\text{def}} \neg(\forall\rho.\neg\gamma).$$

Finally, the set \mathcal{A} of assertions (α) is defined through the following BNF:

$$\alpha ::= \gamma \downarrow^c \xi \mid \rho \downarrow^r \xi \mid \xi : \gamma_1 \sqsubseteq^c \gamma_2 \mid \xi : \rho_1 \sqsubseteq^r \rho_2 \mid \xi_1 \preceq \xi_2 \mid \sim \alpha \mid \alpha_1 \wedge \alpha_2.$$

Strict contextual subsumption and contextual equivalence are obviously defined:

$$\begin{aligned} \xi : \gamma_1 \sqsubseteq^r \gamma_2 &=_{\text{def}} \xi : \gamma_1 \sqsubseteq^c \gamma_2 \wedge \sim \xi : \gamma_2 \sqsubseteq^c \gamma_1 \\ \xi : \rho_1 \sqsubseteq^r \rho_2 &=_{\text{def}} \xi : \rho_1 \sqsubseteq^r \rho_2 \wedge \sim \xi : \rho_2 \sqsubseteq^r \rho_1 \\ \xi : \gamma_1 \equiv^c \gamma_2 &=_{\text{def}} \xi : \gamma_1 \sqsubseteq^c \gamma_2 \wedge \xi : \gamma_2 \sqsubseteq^c \gamma_1 \\ \xi : \rho_1 \equiv^r \rho_2 &=_{\text{def}} \xi : \rho_1 \sqsubseteq^r \rho_2 \wedge \xi : \rho_2 \sqsubseteq^r \rho_1. \end{aligned}$$

The set of atomic assertions of the language is then constituted by expressions enabling exactly the kind of expressivity required in Section 2.2: meaningfulness of concepts and roles in contexts, contextual subsumptions of concepts and roles, generality ordering between contexts.

3.2 Semantics

As exposed in the previous section, a \mathcal{L}^{CT} consists of four classes of expressions: Ξ (context constructs), P and Γ (role and concept descriptions), \mathcal{A} (assertions). Semantics of P and Γ will be the standard description logic semantics of roles and concepts, on which our framework is based. Semantics for Ξ will be given in terms of model theoretic operations on sets of description logic models, and at that stage the semantics of assertions \mathcal{A} will be defined via an appropriate satisfaction relation. The structures obtained, which we call *contextual terminology models* or ct-models, provides a formal semantics for \mathcal{L}^{CT} languages.

The first step is then to provide the definition of a description logic model for a language \mathcal{L}_i ([2]).

Definition 1. (Models for \mathcal{L}_i 's)

A model m for a language \mathcal{L}_i is defined as follows:

$$m = \langle \Delta_m, \mathcal{I}_m \rangle$$

where:

- Δ_m is the (non empty) domain of the model;
- \mathcal{I}_m is a function $\mathcal{I}_m : \mathbf{A}_i \cup \mathbf{R}_i \longrightarrow \mathcal{P}(\Delta_m) \cup \mathcal{P}(\Delta_m \times \Delta_m)$, such that to every element of \mathbf{A}_i and \mathbf{R}_i an element of $\mathcal{P}(\Delta_m)$ and, respectively, of $\mathcal{P}(\Delta_m \times \Delta_m)$ is associated. This interpretation of atomic concepts and roles of \mathcal{L}_i on Δ_m is then inductively extended:

$$\begin{aligned}
\mathcal{I}_m(\top) &= \Delta_m \\
\mathcal{I}_m(\perp) &= \emptyset \\
\mathcal{I}_m(\neg\gamma) &= \Delta_m \setminus \mathcal{I}_m(\gamma) \\
\mathcal{I}_m(\gamma_1 \sqcap \gamma_2) &= \mathcal{I}_m(\gamma_1) \cap \mathcal{I}_m(\gamma_2) \\
\mathcal{I}_m(\forall\rho.\gamma) &= \{a \in \Delta_m \mid \forall b, \langle a, b \rangle \in I_m(\rho) \Rightarrow b \in I_m(\gamma)\} \\
\mathcal{I}_m(\bar{\rho}) &= \Delta_m \times \Delta_m \setminus \mathcal{I}_m(\rho).
\end{aligned}$$

A model m for a language \mathcal{L}_i assigns a denotation to each atomic concept (for instance the set of elements of Δ_m that instantiate the concept **bike**) and to each atomic role (for instance the set of pairs of Δ_m which are in a relation such that the first element is said to “drive” the second element of the pair). Accordingly, meaning is given to each complex concept (for instance the set of elements of Δ_m that instantiate the concept **vehicle** \sqcup **bike**) and to each complex role (for instance the set of pairs listing elements related by role **drive**).

3.3 Models for \mathcal{L}^{CT}

We can now define a notion of *contextual terminology model* (ct-model) for languages \mathcal{L}^{CT} .

Definition 2. (ct-models)

A ct-model \mathbb{M} is a structure:

$$\mathbb{M} = \langle \{\mathbf{M}_i\}_{0 \leq i \leq n}, \mathbb{I} \rangle$$

where:

- $\{\mathbf{M}_i\}_{0 \leq i \leq n}$ is the family of the sets of models \mathbf{M}_i of each language \mathcal{L}_i . That is, $\forall m \in \mathbf{M}_i$, m is a model for \mathcal{L}_i .
- \mathbb{I} is a function $\mathbb{I} : \mathbf{c} \longrightarrow \mathcal{P}(\mathbf{M}_0) \cup \dots \cup \mathcal{P}(\mathbf{M}_n)$. In other words, this function associates to each atomic context identifier in \mathbf{c} a subset of the set of all models in some language \mathcal{L}_i : $\mathbb{I}(c) = M$ with $M \subseteq \mathbf{M}_i$ for some i s.t. $0 \leq i \leq n$. Function \mathbb{I} can be seen as labeling sets of models on some language i via atomic context identifiers. Notice that \mathbb{I} fixes, for each atomic context identifier, the language on which the context denoted by the identifier is specified. We could say that it is \mathbb{I} itself which fixes a specific index for each atomic context identifier c .
- $\forall m', m'' \in \bigcup_{0 \leq i \leq n} \mathbf{M}_i$, $\Delta_{m'} = \Delta_{m''}$. That is, the domain of all models m is unique. We assume this constraint simply because we are interested in modeling different conceptualizations of a same set of individuals.

Contexts are therefore formalized as sets of models for the same language. This perspective allows for straightforward model theoretical definitions of operations on contexts.

3.4 Context focus

We model focus as a specific operation on sets of models which provides the semantic counterpart for the *contextual focus* operator introduced in \mathcal{L}^{CT} . Intuitively, abstracting a context ξ to a language \mathcal{L}_i yields a context consisting in that part of ξ which can be expressed in \mathcal{L}_i .

Let us first recall a notion of *domain restriction* (\lrcorner) of a function f w.r.t. a subset C of the domain of f . Intuitively, a domain restriction of a function f is nothing but the function $C \lrcorner f$ having C as domain and s.t. for each element of C , f and $C \lrcorner f$ return the same image: $C \lrcorner f = \{(x, f(x)) \mid x \in C\}$.

Definition 3. (Context focus operation: \lrcorner_i)

Let M' be a set of models, then: $\lrcorner_i M' = \{m \mid m = \langle \Delta_{m'}, \mathbf{A}_i \cup \mathbf{R}_i \lrcorner \mathcal{I}_{m'} \rangle \ \& \ m' \in M'\}$.

The following can be proved.

Proposition 1. (Properties of context focus)

Operation \lrcorner_i is: *surjective*, *idempotent* ($\lrcorner_i(\lrcorner_i M) = \lrcorner_i M$), *normal* ($\lrcorner_i \emptyset = \emptyset$), *additive* ($\lrcorner_i(M_1 \cup M_2) = \lrcorner_i M_1 \cup \lrcorner_i M_2$), *monotonic* ($M_1 \subseteq M_2 \Rightarrow \lrcorner_i M_1 \subseteq \lrcorner_i M_2$).

Proof. A proof is worked out in [11].

The operation of focus allows for shifting from richer to simpler languages and it is, as we would intuitively expect: surjective (every context, even the empty one, can be seen as the result of focusing a different richer context, in the most trivial case, a focus of itself), idempotent (focusing on a focus yields the same first focus), normal (focusing the empty context yields the empty context), additive (the focus of a context obtained via joining of two contexts can be obtained also joining the focuses of the two contexts), monotonic (if a context is less general than another one, the focus of the first is also less general than the focus of the second one). Notice also that operation \lrcorner_i yields the empty set of models when it is applied to a context M' the language of which is not an expansion of \mathcal{L}_i . This is indeed very intuitive: the context obtained via focus of the context “dinosaurs” on the language of, say, “gourmet cuisine” should be empty.

A detailed comparison of our account of focus with approaches available in the literature on context theory is discussed in [11].

3.5 Operations on contexts

We are now in a position to give a semantics to context constructs as introduced in Section 3.1. In Definition 2 atomic contexts are interpreted as sets of models on some language \mathcal{L}_i for $0 \leq i \leq n$: $\mathbb{I}(c) = M \in \mathcal{P}(\mathbf{M}_0) \cup \dots \cup \mathcal{P}(\mathbf{M}_n)$. The semantics of context constructs Ξ can be defined via inductive extension of that definition.

Definition 4. (Semantics of context constructs)

Let ξ, ξ_1, ξ_2 be context constructs, then:

$$\begin{aligned} \mathbb{I}(\text{fcs}_i \xi) &= \downarrow_i \mathbb{I}(\xi) \\ \mathbb{I}(\perp_i) &= \emptyset \\ \mathbb{I}(\top_i) &= \mathbf{M}_i \\ \mathbb{I}(\xi_1 \gamma_i \xi_2) &= \downarrow_i (\mathbb{I}(\xi_1) \cup \mathbb{I}(\xi_2)). \end{aligned}$$

The focus operator fcs_i is interpreted on the contextual focus operation introduced in Definition 3, i.e., as the restriction of the interpretation of its argument to language \mathcal{L}_i . The \perp_i context is interpreted as the empty context (the same on each language); the \top_i context is interpreted as the greatest, or most general, context on \mathcal{L}_i ; the binary γ_i -composition of contexts is interpreted as the lowest upper bound of the restriction of the interpretations of the two contexts on \mathcal{L}_i .

In [15] the statement about the need for addressing “contexts as abstract mathematical entities” was set forth. Here, moving from an analysis of contextual terminologies, we develop an account of context interplay based on model theoretic operations. In some sense, we propose a view on contexts as “algebraic entities”. In fact, it is easy to prove ([11]) that contexts, as conceived here, are structured according to a Boolean Algebra with Operators ([14]). This observation distills the type of conception of context we hold here: contexts are sets of models on different concept description languages; on each language the set of possible contexts is structured in a Boolean Algebra; adding operations of focus on a finite number of sublanguages yields a Boolean Algebra with Operators.

3.6 Assertions

The semantics of assertions is defined as follows.

Definition 5. (Semantics of assertions: \models)

Let ξ, ξ_1, ξ_2 be a context constructs, $\gamma, \gamma_1, \gamma_2$ concept description, then:

$$\mathbb{M} \models \gamma \downarrow \xi \quad \text{iff} \quad \{D_c \mid \langle \gamma, D_c \rangle \in \mathcal{I}_m \ \& \ m \in \mathbb{I}(\xi)\} \neq \emptyset \quad (1)$$

$$\mathbb{M} \models \rho \downarrow \xi \quad \text{iff} \quad \{D_r \mid \langle \rho, D_r \rangle \in \mathcal{I}_m \ \& \ m \in \mathbb{I}(\xi)\} \neq \emptyset \quad (2)$$

$$\begin{aligned} \mathbb{M} \models \xi : \gamma_1 \sqsubseteq \gamma_2 \quad \text{iff} \quad & \mathbb{M} \models \gamma_1 \downarrow \xi, \mathbb{M} \models \gamma_2 \downarrow \xi \\ & \text{and } \forall m \in \mathbb{I}(\xi) \ \mathcal{I}_m(\gamma_1) \subseteq \mathcal{I}_m(\gamma_2) \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbb{M} \models \xi : \rho_1 \sqsubseteq \rho_2 \quad \text{iff} \quad & \mathbb{M} \models \rho_1 \downarrow \xi, \mathbb{M} \models \rho_2 \downarrow \xi \\ & \text{and } \forall m \in \mathbb{I}(\xi) \ \mathcal{I}_m(\rho_1) \subseteq \mathcal{I}_m(\rho_2) \end{aligned} \quad (4)$$

$$\mathbb{M} \models \xi_1 \preceq \xi_2 \quad \text{iff} \quad \mathbb{I}(\xi_1) \subseteq \mathbb{I}(\xi_2). \quad (5)$$

Clauses (1) and (2) specify when a concept, respectively a role, is meaningful with respect to a context. This is the case when the set of denotations D_c and D_r which the models constituting the context attribute to that concept (D_c being a set of elements of the domain) or that role (D_r being a set of pairs of elements of the domain), is not empty. If concept γ is not expressible in the

language of context ξ , then concept γ gets no denotation at all in context ξ . This happens simply because concept γ does not belong to the domain of functions \mathcal{I}_m , and there therefore exists no interpretation for that concept in the models constituting ξ . The same holds for a role ρ . Clauses (3) and (4) deal with satisfaction of contextual subsumptions. A contextual concept subsumption relation between γ_1 and γ_2 holds iff concepts γ_1 and γ_2 are defined in the models constituting context ξ , i.e., they receive a denotation in those models, and all the description logic models constituting that context interpret γ_1 as a subconcept of γ_2 . Note that this is precisely the clause for the validity of a subsumption relation in standard description logics, but together with the fact that the concepts involved are actually meaningful in that context. Intuitively, we interpret contextual subsumption relations as inherently presupposing the meaningfulness of their terms¹⁰. A perfectly analogous observation holds also for the clause regarding contextual role subsumption relations. Clause (5) gives a semantics to the \preceq relation between context constructs interpreting it as a standard subset relation: $\xi_1 \preceq \xi_2$ means that context denoted by ξ_1 contains at most all the models that ξ_2 contains, that is to say, ξ_1 is *at most as general as* ξ_2 . Clauses for boolean connectives are the obvious ones.

Notions of validity and logical consequence are classically defined. An assertion α is valid if all ct-models \mathbb{M} satisfy it. An assertion α is a logical consequence of the set of assertions $\alpha_1, \dots, \alpha_n$ if all ct-models satisfying $\alpha_1, \dots, \alpha_n$ satisfy α .

4 Contextual Terminologies at Work

4.1 Formalizing the first scenario

We are now in the position to formalize Example 1.

Example 3. (Sufficient conditions for “liability”) To formalize the first scenario within our setting a language \mathcal{L} is needed, which contains the following atomic concepts: `person`, `liable`, `vehicle`, `car`, `bicycle`; and the following atomic roles: `drive` and `wheel`. Four atomic contexts are at issue here: the context of the main regulation R , let us call it c_R ; the contexts of the municipal regulations $M1$, $M2$ and $M3$, let us call them c_{M1} , c_{M2} and c_{M3} respectively. These contexts should be interpreted on two relevant languages (let us call them \mathcal{L}_0 and \mathcal{L}_1) s.t. $\mathbf{A}_0 = \{\text{person, liable, vehicle}\}$, $\mathbf{R}_0 = \{\text{drive}\}$ and $\mathbf{A}_1 = \{\text{person, liable, vehicle, car, bicycle}\}$, $\mathbf{R}_1 = \{\text{drive, wheel}\}$. That is to say, an abstract language concerning only persons, liability, vehicles and the action of driving, and a more detailed language concerning, besides liable persons, vehicles and driving, also cars, bicycles and the action of wheeling. The sets of all models for \mathcal{L}_0 and \mathcal{L}_1 are then respectively \mathbf{M}_0 and \mathbf{M}_1 .

¹⁰ For a more detailed discussion of these clauses we refer the reader to [12].

To model the desired situation, our ct-model should then at least satisfy the following \mathcal{L}^{CT} formulas:

$$c_{M1} \Upsilon_0 c_{M2} \Upsilon_0 c_{M3} \preceq c_R \quad (6)$$

$$\sim (\text{car} \downarrow c_R) \wedge \sim (\text{bicycle} \downarrow c_R) \wedge \sim (\text{wheel} \downarrow c_R) \quad (7)$$

$$c_R : \text{person} \sqcap \exists \text{drive.vehicle} \sqsubseteq \text{person} \sqcap \text{liable} \quad (8)$$

$$c_{M1} \Upsilon_1 c_{M2} \Upsilon_1 c_{M3} : \text{car} \sqsubseteq \text{vehicle} \quad (9)$$

$$c_{M1} \Upsilon_1 c_{M2} : \text{bicycle} \sqsubseteq \text{vehicle} \quad (10)$$

$$c_{M3} : \text{bicycle} \sqsubseteq \neg \text{vehicle} \quad (11)$$

$$c_{M1} \Upsilon_1 c_{M3} : \text{wheel} \sqsubseteq \text{drive} \quad (12)$$

$$c_{M2} : \text{wheel} \sqsubseteq \overline{\text{drive}} \quad (13)$$

$$c_{M1} \Upsilon_1 c_{M2} \Upsilon_1 c_{M3} : \exists \text{wheel.car} \equiv \perp \quad (14)$$

Formula (6) plays a key role, stating that the three contexts c_{M1} , c_{M2} , c_{M3} are concrete variants of context c_R . It tells this by saying that the context obtained by joining the three concrete contexts on language \mathcal{L}_0 (the language of c_R) is at most as general as context c_R , that is: $\uparrow_0 \mathbb{I}(c_{M1}) \cup \uparrow_0 \mathbb{I}(c_{M2}) \cup \uparrow_0 \mathbb{I}(c_{M3}) \subseteq \mathbb{I}(c_R)$ (see Section 3.2). As we will see in the following, this makes c_{M1} , c_{M2} and c_{M3} inherit what holds in c_R . Formula (7) specifies what concepts and roles do not get interpretation in the abstract context c_R . Formula (8) formalizes the abstract rule to the effect that persons driving vehicles (within public parks) are liable for a violation of the applicable regulation. Formulas (9)-(11) describe the different taxonomies holding in the three concrete contexts at issue, while formulas (12) and (13) describe the different role hierarchies holding in those contexts. The last formula can be seen as simply stating some background knowledge to the effect that to wheel a car is an empty concept.

To discuss in some more depth the proposed formalization, let us first list some interesting logical consequences of formulas (6)-(14). We will focus on assumptions contextualized to monadic contexts, that is to say, we will show what the consequences of formulas (6)-(14) are at the level of the three contexts c_{M1} , c_{M2} and c_{M3} considered in isolation.

$$(6, 8) \models c_{M1} : \text{person} \sqcap \exists \text{drive.vehicle} \sqsubseteq \text{person} \sqcap \text{liable}$$

$$(9) \models c_{M1} : \text{person} \sqcap \exists \text{drive.car} \sqsubseteq \text{person} \sqcap \exists \text{drive.vehicle}$$

$$(10) \models c_{M1} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \exists \text{drive.vehicle}$$

$$(10, 12) \models c_{M1} : \text{person} \sqcap \exists \text{wheel.bicycle} \sqsubseteq \text{person} \sqcap \exists \text{drive.bicycle}$$

$$(6, 8, 10) \models c_{M1} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \text{liable}$$

$$(6, 8, 10, 12) \models c_{M1} : \text{person} \sqcap \exists \text{wheel.bicycle} \sqsubseteq \text{person} \sqcap \text{liable}$$

(6, 8) $\models c_{M_2} : \text{person} \sqcap \exists \text{drive.vehicle} \sqsubseteq \text{person} \sqcap \text{liable}$
 (9) $\models c_{M_2} : \text{person} \sqcap \exists \text{drive.car} \sqsubseteq \text{person} \sqcap \exists \text{drive.vehicle}$
 (10) $\models c_{M_2} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \exists \text{drive.vehicle}$
 (13) $\models c_{M_2} : \text{person} \sqcap \exists \text{wheel.bicycle} \sqsubseteq \text{person} \sqcap \overline{\exists \text{drive.bicycle}}$
 (10, 13) $\models c_{M_2} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \overline{\exists \text{drive.vehicle}}$
 (6, 8, 10) $\models c_{M_2} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \text{liable}$
 (6, 8) $\models c_{M_2} : \text{person} \sqcap \neg \text{liable} \sqsubseteq \text{person} \sqcap \neg \exists \text{drive.vehicle}$
 (6, 8, 9, 10, 13) $\models \sim c_{M_2} : \text{person} \sqcap \exists \text{wheel.bicycle} \sqsubseteq \text{person} \sqcap \text{liable}$

(6, 8) $\models c_{M_3} : \text{person} \sqcap \exists \text{drive.vehicle} \sqsubseteq \text{person} \sqcap \text{liable}$
 (9) $\models c_{M_3} : \text{person} \sqcap \exists \text{drive.car} \sqsubseteq \text{person} \sqcap \exists \text{drive.vehicle}$
 (11) $\models c_{M_3} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \exists \text{drive.}\neg \text{vehicle}$
 (12) $\models c_{M_3} : \text{person} \sqcap \exists \text{wheel.bicycle} \sqsubseteq \text{person} \sqcap \exists \text{drive.bicycle}$
 (6, 8) $\models c_{M_3} : \text{person} \sqcap \neg \text{liable} \sqsubseteq \text{person} \sqcap \neg \exists \text{drive.vehicle}$
 (6, 8, 11) $\models \sim c_{M_3} : \text{person} \sqcap \exists \text{drive.bicycle} \sqsubseteq \text{person} \sqcap \text{liable}$
 (6, 8, 11, 12) $\models \sim c_{M_3} : \text{person} \sqcap \exists \text{wheel.bicycle} \sqsubseteq \text{person} \sqcap \text{liable}$

These are indeed the formulas that we would intuitively expect to hold in our scenario. The list displays three sets of formulas grouped on the basis of the context to which they pertain. Let us have a closer look to them. The first consequence of each group results from the generality relation expressed in (6), by means of which, the content of (8) is shown to hold also in the three concrete contexts: in simple words, contexts c_{M_1} , c_{M_2} and c_{M_3} inherit the general rule stating the liability of persons driving vehicles (within public parks). Via this inherited rule, and via (9), it is shown that, in all contexts, who drives a car is also held liable (second consequence of each group). As to cars and driving cars then, all contexts agree. Where differences arise is in relation with how the concept of bicycle and the role of wheeling are handled.

In context c_{M_1} , we have that it does not matter if somebody wheels or actually drives a bicycle, because in both cases this would count as driving a vehicle, and therefore of violating the regulation. In fact, in this context, a bicycle is a vehicle (10) and to wheel is a way of driving (11). Context c_{M_2} , instead, expresses a different view. Since bicycles count as vehicles (10), to drive a bicycle is still a ground for liability. On the other hand, to wheel is actually classified as a way of refraining from driving (13), and therefore, persons wheeling bicycles do not count as persons driving vehicles, and do not commit any violation. Context c_{M_3} yields yet another terminology. Here bicycles are classified as objects which are not vehicles (11). Therefore, although to wheel is conceived as a way of driving (11), both to drive and to wheel a bicycle does not determine liability. With respect to this, it is instructive to notice that even though both in c_{M_2} and c_{M_3} to wheel a bicycle is not a sufficient reason for being held liable, this holds for two different reasons: in c_{M_2} because of (13), and in c_{M_3} because of

(11). This illustrates how our framework is able to cope with some quite subtle nuances that characterize contextual classifications.

4.2 A model of the scenario

In this section we expose a simple ct-model satisfying (6)-(14). Let us stipulate that the models m that constitute our interpretation of contexts identifiers consist of a domain $\Delta_m = \{a, b, c, d, e, f, g\}$. Being \mathcal{L}_0 and \mathcal{L}_1 the two languages at issue, the domain of the ct-models is $\mathbf{M}_0 \cup \mathbf{M}_1$. A ct-model would then be, for instance, a structure $\langle \mathbf{M}_0 \cup \mathbf{M}_1, \mathbb{I} \rangle$ where \mathbb{I} is such that:

- $\mathbb{I}(c_{M1}) = \{m_1, m_2\} \subseteq \mathbf{M}_1$ s.t. $\mathcal{I}_{m_1}(\mathbf{person}) = \{e, f, g\}$, $\mathcal{I}_{m_1}(\mathbf{vehicle}) = \{a, b, c, d\}$, $\mathcal{I}_{m_1}(\mathbf{bicycle}) = \{a, b\}$, $\mathcal{I}_{m_1}(\mathbf{car}) = \{c, d\}$, $\mathcal{I}_{m_1}(\mathbf{drive}) = \{\langle e, a \rangle, \langle f, c \rangle\}$, $\mathcal{I}_{m_1}(\mathbf{wheel}) = \{\langle e, a \rangle\}$, $\mathcal{I}_{m_1}(\mathbf{liable}) = \{e, f\}$ and \mathcal{I}_{m_2} agrees with \mathcal{I}_{m_1} on the interpretation of **person**, **bicycle**, **car**, **vehicle** and $\mathcal{I}_{m_2}(\mathbf{drive}) = \{\langle f, c \rangle, \langle g, d \rangle\}$, $\mathcal{I}_{m_2}(\mathbf{wheel}) = \{\langle g, d \rangle\}$, $\mathcal{I}_{m_2}(\mathbf{liable}) = \{f, g\}$.
- $\mathbb{I}(c_{M2}) = \{m_3, m_4\} \subseteq \mathbf{M}_1$ s.t. \mathcal{I}_{m_3} and \mathcal{I}_{m_4} agree with \mathcal{I}_{m_1} on the interpretation of **person**, **bicycle**, **car**, **vehicle** and $\mathcal{I}_{m_3}(\mathbf{drive}) = \{\langle f, d \rangle, \langle g, a \rangle\}$, $\mathcal{I}_{m_3}(\mathbf{wheel}) = \{\langle e, a \rangle\}$, $\mathcal{I}_{m_3}(\mathbf{liable}) = \{f, g\}$ and $\mathcal{I}_{m_4}(\mathbf{drive}) = \{\langle e, c \rangle\}$, $\mathcal{I}_{m_4}(\mathbf{wheel}) = \{\langle f, a \rangle\}$, $\mathcal{I}_{m_4}(\mathbf{liable}) = \{e\}$.
- $\mathbb{I}(c_{M3}) = \{m_5\} \subseteq \mathbf{M}_1$ s.t. \mathcal{I}_{m_5} agrees with \mathcal{I}_{m_1} on the interpretation of **person**, **bicycle**, **car** and $\mathcal{I}_{m_5}(\mathbf{vehicle}) = \{c, d\}$, $\mathcal{I}_{m_5}(\mathbf{drive}) = \{\langle e, a \rangle, \langle f, c \rangle, \langle g, d \rangle\}$, $\mathcal{I}_{m_5}(\mathbf{wheel}) = \{\langle e, a \rangle\}$, $\mathcal{I}_{m_5}(\mathbf{liable}) = \{f, g\}$.
- $\mathbb{I}(c_R) = \{m \mid m = \langle \Delta_m, \mathbf{A}_0 \cup \mathbf{R}_0 \rangle \mathcal{I}_i\}$ and $1 \leq i \leq 5$, that is, c_R is interpreted by the model as the union of all models constituting c_{M1} , c_{M2} and c_{M3} restricted to the language \mathcal{L}_0 .

The model makes an interesting feature of our semantics explicit. In contexts c_{M1} and c_{M2} the set of liable persons do not coincide in the two models constituting the context; nevertheless only persons driving vehicles are indeed liable. This clearly shows that contexts can be viewed as clusters of possible situations all instantiating the same terminology¹¹.

4.3 Formalizing the second scenario

The formalization of the scenario introduced in Example 2 follows.

Example 4. (Categorizing “teenagers on skates”) The global language \mathcal{L} needed contains the following atomic concepts: **conv**, **person**, **obj**, **vehicle**, **teenager**, **skate**; and the following atomic role: **transp**. Three are the atomic contexts at issue here: the context of the main regulation R, let us call it c_R ; the contexts of the municipal regulations M1 and M2, let us call them c_{M1} and c_{M2} respectively. These contexts should be interpreted on two relevant languages (let

¹¹ We developed this intuition also in a modal logic setting modeling contexts as sets of possible worlds. See [13].

us call them \mathcal{L}_0 and \mathcal{L}_1) s.t. $\mathbf{A}_0 = \{\text{conv}, \text{person}, \text{obj}, \text{vehicle}\}$, $\mathbf{R}_0 = \{\text{transp}\}$ and $\mathbf{A}_1 = \mathbf{A}_0 \cup \{\text{self_prop}, \text{teenager}, \text{skate}\}$, $\mathbf{R}_1 = \mathbf{R}_0$. That is to say, an abstract language concerning only conveyances, persons, objects, vehicles and the attribute of transporting, and a more detailed language concerning, besides this, also teenagers and skates. The sets of all models for \mathcal{L}_0 and \mathcal{L}_1 are then respectively \mathbf{M}_0 and \mathbf{M}_1 . To model the desired situation, a ct-model should then at least satisfy the following \mathcal{L}^{CT} formulas:

$$c_{M1} \Upsilon_0 c_{M2} \preceq c_R \quad (15)$$

$$\sim(\text{teenager} \downarrow c_R) \wedge \sim(\text{skate} \downarrow c_R) \quad (16)$$

$$c_R : \text{vehicle} \sqsubseteq \text{conv} \sqcap \forall \text{transp.}(\text{person} \sqcup \text{obj}) \quad (17)$$

$$c_{M1} : \text{vehicle} \sqsubseteq \text{self_prop} \quad (18)$$

$$c_{M1} : \text{conv} \sqcap \forall \text{transp.}(\text{person} \sqcup \text{obj}) \sqcap \text{self_prop} \sqsubseteq \text{vehicle} \quad (19)$$

$$c_{M2} : \text{conv} \sqcap \forall \text{transp.}(\text{person} \sqcup \text{obj}) \sqsubseteq \text{vehicle} \quad (20)$$

$$c_{M1} \Upsilon_1 c_{M2} : \text{teenager} \sqsubseteq \text{person} \quad (21)$$

$$c_{M1} \Upsilon_1 c_{M2} : \text{skate} \sqsubseteq \text{conv} \quad (22)$$

$$c_{M1} \Upsilon_1 c_{M2} : \text{skate} \sqsubseteq \neg \text{self_prop} \quad (23)$$

For reason of space, we cannot discuss this example in as many details as the previous one. We stress the most important aspects. Formulas (15) and (16) are the analogous of formulas (6) and (7). Formula (17) represents the abstract constraints that context c_R imposes on the concept **vehicle**.

Formulas (18), (19) and (20) express the additional constraints on the concept **vehicle** holding in context c_{M1} and c_{M2} respectively: both contexts specify sufficient conditions and context c_{M1} adds also new necessary ones (18). Formulas (21) and (22) state the intuitive background knowledge common to the two concrete contexts. The point of the scenario consists in showing how teenagers on skateboards are conceptualized in the three contexts, that is to say: how are concept $\text{skate} \sqcap \exists \text{transp.} \text{teenager}$ and concept **vehicle** related in each context? This can be easily shown via some relevant logical consequences of (15)-(23):

$$(15, 17, 18, 19) \models c_{M1} : \text{conv} \sqcap \forall \text{transp.}(\text{person} \sqcup \text{obj}) \sqcap \text{self_prop} \equiv \text{vehicle}$$

$$(15, 17, 18, 19, 21, 22, 23) \models c_{M1} : \text{skate} \sqcap \exists \text{transp.} \text{teenager} \sqsubseteq \neg \text{vehicle}$$

$$(15, 17, 20) \models c_{M2} : \text{conv} \sqcap \forall \text{transp.}(\text{person} \sqcup \text{obj}) \equiv \text{vehicle}$$

$$(15, 17, 20, 21, 22) \models c_{M2} : \text{skate} \sqcap \exists \text{transp.} \text{teenager} \sqsubseteq \text{vehicle}.$$

These formulas show that in the two concrete contexts two different definitions of **vehicle** hold, and therefore two different conceptualizations of concepts such as $\text{skate} \sqcap \exists \text{transp.} \text{teenager}$: since skateboards are, in c_{M1} , non self-propelled, even if they are conveyances transporting people, they are not classifiable as vehicles .

5 Conclusions

We motivated and devised a formal framework for representing contextual ontologies via a contextualized version of description logic semantics. The key idea has been to show that the basic intuition of understanding contexts as sets of description logic models, which we presented in [12], works smoothly also with subsumption statements of more complex concept descriptions. The next step will be to side contextual terminologies with appropriate contextual assertion boxes (ABoxes) in which to reason about contextual instantiations of concepts and roles.

In future work we will focus on developing a proof theory along the lines sketched in [11] and on analyzing the complexity of the framework especially with respect to inter-contextual reasoning. We intend also to apply contextual terminologies to the study of the contextual meaning of actions in agents institutions: raising a hand during a bid has a different meaning than raising a hand during a scientific workshop. A natural way to do this, would be to exploit established results about the relation between dynamic logic and description logic ([4]) to get to a contextualized form of dynamic logic.

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