# Strongly Complete Axiomatizations of “Knowing At Most” in Standard Syntactic Assignments

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**Abstract.** Standard syntactic assignments (SSAs) model knowledge directly rather than as truth in all possible worlds as in modal epistemic logic, by assigning arbitrary truth values to atomic epistemic formulae. It is a very general approach to epistemic logic, but has no interesting logical properties — partly because the standard logical language is too weak to express properties of such structures. In this paper we extend the logical language with a new operator used to represent the proposition that an agent “knows at most” a given finite set of formulae and study the problem of strongly complete axiomatization of SSAs in this language. Since the logic is not semantically compact, a strongly complete *finitary* axiomatization is impossible. Instead we present, first, a strongly complete *infinitary* system, and, second, a strongly complete finitary system for a slightly weaker variant of the language.

## 1 Introduction

In traditional modal epistemic logic [1, 2], modelling knowledge as truth in all possible states in a Kripke structure, agents know all the logical consequences of their knowledge. It fails as a logic of the *computed* knowledge of real agents, because it assumes a very particular and extremely powerful mechanism for reasoning. In reality, different agents have different reasoning mechanisms (e.g. non-monotonic or resource-bounded) and representations of knowledge (e.g. as propositions or as syntactic formulae). Thus, a more general model of knowledge would be useful. A general approach is to model knowledge *directly* rather than as truth in all possible worlds. *Standard Syntactic Assignments (SSAs)* [1] is a syntactic approach in which a formula $K_i\phi$ is assigned a truth value independent of the truth value assigned to any other formula of the form $K_i\psi$. SSAs are generalizations of Kripke structures. In fact, it can be argued that SSAs is the *most* general model of knowledge. However, SSAs are so general that they have no interesting logical properties that can be expressed in the traditional language of epistemic logic – indeed, they are completely axiomatized by propositional logic.

In this paper, we extend the logical language with a new epistemic operator $\nabla_i$ for each agent. $\nabla_i X$, where $X$ is a finite set of formulae, expresses the fact that agent $i$ knows *at most* $X$. The main problem we consider is the construction of a strongly complete axiomatization of SSAs in this language. A consequence of the addition of the new operator is that semantic compactness is lost, and thus that a strongly complete finitary axiomatization is impossible. Instead we, first, present a strongly complete *infinitary* system, and, second, a strongly complete finitary system for SSAs for a slightly weaker variants of the epistemic operators. The results are a contribution to the logical foundation of multi-agent systems.

In Section 2 SSAs and their use in epistemic logic are introduced, before the "at most" operator $\nabla_i$ and its interpretation in SSAs is presented in Section 3. The completeness results are presented in Section 4, and we conclude and discuss related work in Section 5. We presently define some logical concepts and terminology used in the remainder.

### 1.1 Logic

By "a logic" we henceforth mean a language of formulae together with a satisfiability relation $\models$. The semantic structures considered in this paper each has a set of *states*, and satisfiability relations relate a formula to a pair consisting of a structure $M$ and a state $s$ of $M$. A formula $\phi$ is a (local) *logical consequence* of a theory (set of formulae) $\Gamma$, $\Gamma \models \phi$, iff $(M, s) \models \phi$ for all $\phi \in \Gamma$ implies that $(M, s) \models \phi$. The usual terminology and notation for Hilbert-style proof systems are used: $\Gamma \vdash_S \phi$ means that formula $\phi$ is derivable from theory $\Gamma$ in system $S$, and when $\Delta$ is a set of formulae, $\Gamma \vdash_S \Delta$ means that $\Gamma \vdash_S \delta$ for each $\delta \in \Delta$. We use the following definition of maximality: a theory in a language $L$ is maximal if it contains either $\phi$ or $\neg\phi$ for each $\phi \in L$. A logical system is *weakly complete*, or just *complete*, if $\models \phi$ (i.e. $\emptyset \models \phi$, $\phi$ is *valid*) implies $\vdash_S \phi$ (i.e. $\emptyset \vdash_S \phi$) for all formulae $\phi$, and *strongly complete* if $\Gamma \models \phi$ implies $\Gamma \vdash_S \phi$ for all formulae $\phi$ and theories $\Gamma$. If a logic has a (strongly) complete logical system, we say that the logic *is* (strongly) complete. A logic is semantically *compact* if for every theory $\Gamma$, if every finite subset of $\Gamma$ is satisfiable then $\Gamma$ is satisfiable. It is easy to see that under the definitions used above:

**Fact 1** A weakly complete logic has a sound and strongly complete finitary axiomatization iff it is compact.

## 2 The Epistemic Logic of Standard Syntactic Assignments

Standard Syntactic Assignments (SSAs) are defined and interpret the standard epistemic language, as follows. Given a number of agents $n$ we write $\Sigma$ for the set $\{1, \ldots, n\}$.

**Definition 2 ($\mathcal{L}$)** Given a set of primitive propositions $\Theta$ and a number of agents $n$, $\mathcal{L}(\Theta, n)$ (or just $\mathcal{L}$) is the least set such that:

- $\Theta \subseteq \mathcal{L}$
- If $\phi, \psi \in \mathcal{L}$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{L}$
- If $\phi \in \mathcal{L}$ and $i \in \Sigma$ then $K_i\phi \in \mathcal{L}$ □

The set of *epistemic atoms* is $\mathcal{L}^{At} = \{K_i\phi : \phi \in \mathcal{L}, i \in \Sigma\}$. An epistemic formula is a propositional combination of epistemic atoms. An SSA assigns a truth value to the primitive propositions and epistemic atoms.

**Definition 3 (Standard Syntactic Assignment)** A standard syntactic assignment (SSA) is a tuple

$$(S, \sigma)$$

where $S$ is a set of states and

$$\sigma(s) : \Theta \cup \mathcal{L}^{At} \rightarrow \{\mathbf{true}, \mathbf{false}\}$$

for each $s \in S$. □

Satisfaction of an $\mathcal{L}$ formula $\phi$ by a state $s$ of an SSA $M$, written $(M, s) \models \phi$, is defined as follows:

$$\begin{array}{lcl} (M,s) \models p & \Leftrightarrow & \sigma(s)(p) = \mathbf{true} \\ (M,s) \models \neg\phi & \Leftrightarrow & (M,s) \not\models \phi \\ (M,s) \models (\phi \wedge \psi) & \Leftrightarrow & (M,s) \models \phi \text{ and } (M,s) \models \psi \\ (M,s) \models K_i\phi & \Leftrightarrow & \sigma(s)(K_i\phi) = \mathbf{true} \end{array}$$

We note that although [1] define SSAs in a possible worlds framework, the question of satisfaction of $\phi$ in a state $s$ does not depend on any other state $(((S, \sigma), s) \models \phi \Leftrightarrow ((\{s\}, \sigma), s) \models \phi)$.

SSAs are very general descriptions of knowledge – in fact so general that no epistemic properties of the class of all SSAs can be described by the standard epistemic language:

**Theorem 4** Propositional logic, with substitution instances for the language $\mathcal{L}$, is sound and complete with respect to SSAs. □

In the next section we increase the expressiveness of the epistemic language.

## 3 Knowing At Most

The formula $K_i\phi$ denotes that fact that $i$ knows *at least* $\phi$ – he knows $\phi$ but he may know more. We can generalize this to finite sets $X$ of formulae:

$$\triangle_i X \equiv \bigwedge\{K_i\phi : \phi \in X\}$$

representing the fact that $i$ knows at least $X$. The new operator we introduce[3] in this paper is a dual to $\triangle_i$, denoting the fact that $i$ knows *at most* $X$:

$$\triangledown_i X$$

[3] A similar operator is also used in [3] and [4].

denotes the fact that every formula an agent knows is included in $X$, but he may not know all the formulae in $X$. If $\mathcal{L}$ was finite, the operator $\triangledown_i$ could (like $\triangle_i$) be defined in terms of $K_i$:

$$\triangledown_i X = \bigwedge\{\neg K_i\phi : \phi \in \mathcal{L} \setminus X\}$$

But since $\mathcal{L}$ is not finite, $\triangledown_i$ is not definable by $K_i$. We also use a third, derived, epistemic operator: $\Diamond_i X \equiv \triangle_i X \wedge \triangledown_i X$ meaning that the agent knows exactly $X$. The extended language is called $\mathcal{L}_\triangledown$.

**Definition 5 ($\mathcal{L}_\triangledown$)** Given a set of primitive propositions $\Theta$, and a number of agents $n$, $\mathcal{L}_\triangledown(\Theta, n)$ (or just $\mathcal{L}_\triangledown$) is the least set such that:

- $\Theta \subseteq \mathcal{L}_\triangledown$
- If $\phi, \psi \in \mathcal{L}_\triangledown$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{L}_\triangledown$
- If $\phi \in \mathcal{L}$ and $i \in \Sigma$ then $K_i\phi \in \mathcal{L}_\triangledown$
- If $X \in \wp^{fin}(\mathcal{L})$ and $i \in \Sigma$ then $\triangledown_i X \in \mathcal{L}_\triangledown$ $\square$

The language $\mathcal{L}_\triangledown(\Theta, n)$ is defined to express properties of SSAs over the language $\mathcal{L}(\Theta, n)$ (introduced in Section 2), and thus the epistemic operators $K_i$ and $\triangledown_i$ operate on formulae from $\mathcal{L}(\Theta, n)$. We assume that $\Theta$ is countable, and will make use of the fact that it follows that $\mathcal{L}_\triangledown(\Theta, n)$ is (infinitely) countable.

If $X$ is a finite set of $\mathcal{L}_\triangledown$ formulae, we write $\triangle_i X$ as a shorthand for $\bigwedge_{\phi \in X} K_i\phi$. In addition, we use $\Diamond_i X$ for $\triangle_i X \wedge \triangledown_i X$, and the usual derived propositional connectives.

The interpretation of $\mathcal{L}_\triangledown$ in a state $s$ of an SSA $M$ is defined in the same way as the interpretation of $\mathcal{L}$, with the following clause for the new epistemic operator:

$$(M, s) \models \triangledown_i X \quad \Leftrightarrow \quad \{\phi \in \mathcal{L} : \sigma(s)(K_i\phi) = \mathbf{true}\} \subseteq X$$

It is easy to see that

$$(M, s) \models \triangle_i X \quad \Leftrightarrow \quad \{\phi \in \mathcal{L} : \sigma(s)(K_i\phi) = \mathbf{true}\} \supseteq X$$

$$(M, s) \models \Diamond_i X \quad \Leftrightarrow \quad \{\phi \in \mathcal{L} : \sigma(s)(K_i\phi) = \mathbf{true}\} = X$$

### 3.1 Properties

The following schemata, where $X, Y, Z$ range over finite sets of formulae and $\phi$ over single formulae, show some properties of SSAs, in the language $\mathcal{L}_\triangledown$.

| | | |
|---|---|---|
| $\triangle_i \emptyset$ | | E1 |
| $\triangle_i X \wedge \triangle_i Y \rightarrow \triangle_i(X \cup Y)$ | | E2 |
| $(\triangledown_i X \wedge \triangledown_i Y) \rightarrow \triangledown_i(X \cap Y)$ | | E3 |
| $\neg(\triangle_i X \wedge \triangledown_i Y)$ | when $X \not\subseteq Y$ | E4 |
| $(\triangledown_i(Y \cup \{\phi\}) \wedge \neg K_i\phi) \rightarrow \triangledown_i Y$ | | E5 |
| $\triangle_i X \rightarrow \triangle_i Y$ | when $Y \subseteq X$ | **KS** |
| $\triangledown_i X \rightarrow \triangledown_i Y$ | when $X \subseteq Y$ | **KG** |

It is straightforward to prove the following.

**Lemma 6** E1–E5, **KG**, **KS** are valid. □

# 4 Axiomatizations of SSAs

In this section we discuss axiomatizations of standard syntactic assignments in the language $\mathcal{L}_\triangledown$. The following lemma shows that the logic is not compact, and thus it does not have a strongly complete finitary axiomatization (Fact 1).

**Lemma 7** The logic of standard syntactic assignments in the language $\mathcal{L}_\triangledown$ is not compact. □

PROOF Let $p \in \Theta$ and let $\Gamma_1$ be the following $\mathcal{L}_\triangledown$ theory:

$$\Gamma_1 = \{K_i p, \neg \triangledown_i \{p\}\} \cup \{\neg K_i \phi : \phi \neq p\}$$

Let $\Gamma'$ be a finite subset of $\Gamma_1$. Clearly, there exists a $\phi'$ such that $\neg K_i \phi' \notin \Gamma'$. Let $M = (\{s\}, \sigma)$ be such that $\sigma(s)(K_i\phi) = \mathbf{true}$ iff $\phi = p$ or $\phi = \phi'$. It is easy to see that $(M, s) \models \Gamma'$. If there was some $(M', s')$ such that $(M', s') \models \Gamma_1$, then $(M', s') \models \neg \triangledown_i \{p\}$ i.e. there must exist a $\phi \neq p$ such that $\sigma(s)(K_i\phi) = \mathbf{true}$ – which contradicts the fact that $(M', s') \models \neg K_i\phi$ for all $\phi \neq p$. Thus, every finite subset of $\Gamma_1$ is satisfiable, but $\Gamma_1$ is not. ■

We present a strongly complete *in*finitary axiomatization in 4.1. Then, in 4.2, a finitary axiomatization for a slightly weaker language than $\mathcal{L}_\triangledown$ is proven strongly complete for SSAs.

## 4.1 An Infinitary System

We define a proof system $EC^\omega$ for the language $\mathcal{L}_\triangledown$ by using properties presented in Section 3 as axioms, in addition to propositional logic. In addition, $EC^\omega$ contains an infinitary derivation clause **R***. After presenting $EC^\omega$, the rest of the section is concerned with proving its strong completeness with respect to the class of all SSAs. This is done by the commonly used strategy of proving satisfiability of maximal consistent theories. Thus we need an infinitary variant of the Lindenbaum lemma. However, the usual proof of the Lindenbaum lemma for finitary systems is not necessarily applicable to infinitary systems. In order to prove the Lindenbaum lemma for $EC^\omega$, we use the same strategy as [5] who prove strong completeness of an infinitary axiomatization of PDL (with canonical models). In particular, we use the same way of defining the derivability relation by using a weakening rule **W**, and we prove the deduction theorem in the same way by including a cut rule **Cut**.

**Definition 8 ($EC^\omega$)** $EC^\omega$ is a logical system for the language $\mathcal{L}_\triangledown$ having the following axiom schemata

| | | |
|---|---|---|
| All substitution instances of tautologies of propositional calculus | | **Prop** |
| $(\triangledown_i X \wedge \triangledown_i Y) \to \triangledown_i (X \cap Y)$ | | E3 |
| $\neg(\triangle_i X \wedge \triangledown_i Y)$ | when $X \not\subseteq Y$ | E4 |
| $(\triangledown_i (Y \cup \{\gamma\}) \wedge \neg K_i \gamma) \to \triangledown_i Y$ | | E5 |
| $\triangledown_i X \to \triangledown_i Y$ | when $X \subseteq Y$ | **KG** |

The derivation relation $\vdash_{EC^\omega}$ – written $\vdash_\omega$ for simplicity – between sets of $\mathcal{L}_\triangledown$ formulae and single $\mathcal{L}_\triangledown$ formulae is the smallest relation closed under the following conditions:

$$\vdash_\omega \phi \quad \text{when } \phi \text{ is an axiom} \qquad \textbf{Ax}$$

$$\{\phi, \phi \to \psi\} \vdash_\omega \psi \qquad \textbf{MP}$$

$$\bigcup_{j \in J} \{\alpha_j \to \neg K_i \gamma : \gamma \notin X_j\} \vdash_\omega \bigwedge_{j \in J} \alpha_j \to \triangledown_i X \qquad \textbf{R*}$$

$$\text{when } X = \bigcap_{j \in J} X_j \text{ and } X \text{ and } J \text{ are finite}$$

$$\frac{\Gamma \vdash_\omega \phi}{\Gamma \cup \Delta \vdash_\omega \phi} \qquad \textbf{W}$$

$$\frac{\Gamma \vdash_\omega \Delta, \Gamma \cup \Delta \vdash_\omega \phi}{\Gamma \vdash_\omega \phi} \qquad \textbf{Cut}$$

In the above schemata, $X$, $Y$, $Z$, $X_j$ range over sets of $\mathcal{L}$ formulae, $\gamma$ over $\mathcal{L}$ formulae, $\Gamma, \Delta$ over sets of $\mathcal{L}_\triangledown$ formulae, $\phi$, $\psi$, $\alpha_j$ over $\mathcal{L}_\triangledown$ formulae, and $i$ over agents. $J$ is some finite set of indices. □

It is easy to see that E1, E2 and **KS** are derivable in $EC^\omega$.

The use of the weakening rule instead of more general schemas makes inductive proofs easier, but particular derivations can sometimes be more cumbersome. For example:

**Lemma 9**

$$\Gamma \cup \{\phi\} \vdash_\omega \phi \qquad \text{R1}$$

$$\frac{\vdash_\omega \psi \to \phi}{\Gamma \cup \{\psi \vdash_\omega \phi\}} \qquad \text{R2}$$

□

Proof

**R1:** $\{\phi, \phi \to \phi\} \vdash_\omega \phi$ by **MP**; $\vdash_\omega \phi \to \phi$ by **Ax**; $\{\phi\} \vdash_\omega \phi \to \phi$ by **W**; $\{\phi\} \vdash_\omega \phi$ by **Cut** and $\Gamma \cup \{\phi\} \vdash_\omega \phi$ by **W**.

**R2:** Let $\vdash_\omega \psi \to \phi$. By **W**, $\{\psi\} \vdash_\omega \psi \to \phi$; by **MP** $\{\psi, \psi \to \phi\} \vdash_\omega \phi$ and thus $\{\psi\} \vdash_\omega \phi$ by **Cut**. By **W**, $\Gamma \cup \{\psi\} \vdash_\omega \phi$. ■

In order to prove the Lindenbaum lemma, we need the deduction theorem. The latter is shown by first proving the following rule.

**Lemma 10** The following rule of *conditionalization* is admissible in $EC^\omega$.

$$\frac{\Gamma \cup \Delta \vdash_\omega \phi}{\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi} \quad \textbf{Cond}$$

□

Proof The proof is by infinitary induction over the derivation $\Gamma \cup \Delta \vdash_\omega \phi$ (derivations are well-founded). The base cases are **Ax**, **MP** and **R***, and the inductive steps are **W** and **Cut**.

**Ax:** $\Gamma = \Delta = \emptyset$. We must show that $\vdash_\omega \psi \to \phi$ when $\vdash_\omega \phi$. By **W** we get $\phi \to (\psi \to \phi) \vdash_\omega \phi$, then $\phi, \phi \to (\psi \to \phi) \vdash_\omega \psi \to \phi$ is an instance of **MP**, and by **Cut** we get $\phi \to (\psi \to \phi) \vdash_\omega \psi \to \phi$. By **Prop**, $\vdash_\omega \phi \to (\psi \to \phi)$, so by **Cut** once more we get $\vdash_\omega \psi \to \phi$.

**MP:** $\Gamma \cup \Delta = \{\phi', \phi' \to \phi\} \vdash_\omega \phi$. That $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi$ can be shown for each of the four possible combinations of $\Gamma$ and $\Delta$ in a similar way to the **Ax** case.

**R*:** $\phi = \wedge_{j \in J} \alpha_j \to \triangledown_i X$ and $\Gamma \cup \Delta = \cup_{j \in J}\{\alpha_j \to \neg K_i \phi' : \phi' \in \mathcal{L} \setminus X_j\}$ where $J$ is finite and $X = \cap_{j \in J} X_j$ is finite, i.e. there exist for each $j \in J$ sets $Y_j$ and $Z_j$ such that $\mathcal{L} \setminus X_j = Y_j \uplus Z_j$ and

$$\Gamma = \bigcup_{j \in J} \{\alpha_j \to \neg K_i \phi' : \phi' \in Y_j\}$$
$$\Delta = \bigcup_{j \in J} \{\alpha_j \to \neg K_i \phi' : \phi' \in Z_j\}$$

Let

$$\Gamma' = \bigcup_{j \in J} \{(\psi \wedge \alpha_j) \to \neg K_i \phi' : \phi' \in Y_j\}$$
$$\Delta' = \bigcup_{j \in J} \{(\psi \wedge \alpha_j) \to \neg K_i \phi' : \phi' \in Z_j\}$$

$\Gamma' \cup \Delta' = \cup_{j \in J}\{(\psi \wedge \alpha_j) \to \neg K_i \phi' : \phi' \in \mathcal{L} \setminus X_j\}$, and thus $\Gamma' \cup \Delta' \vdash_\omega \gamma'$, where $\gamma' = \wedge_{j \in J}(\psi \wedge \alpha_j) \to \triangledown_i X$, by **R***. By **W**, $\Gamma' \cup \Delta' \cup \Gamma \vdash_\omega \gamma'$. By **Prop**, $\vdash_\omega (\alpha_j \to \neg K_i \phi') \to ((\psi \wedge \alpha_j) \to \neg K_i \phi')$ for each $\alpha_j \to \neg K_i \phi' \in \Gamma$, and by R2 (once for each formula in $\Gamma$) $\Delta' \cup \Gamma \vdash_\omega \Gamma'$. By **Cut**, $\Delta' \cup \Gamma \vdash_\omega \gamma'$, and it only remains to convert the conjunctions in $\Delta'$ and $\gamma'$ to implications: $\Delta' \cup \Gamma \cup \{\gamma'\} \vdash_\omega \psi \to \phi$ by **Prop** and R2, and by **Cut** and **W** it follows that $\Delta' \cup \Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi$. By **Prop** and R2 (once of each formula in $\Delta$), $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \Delta'$, and by **Cut** $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi$, which is the desired conclusion.

**W:** $\Gamma' \cup \Delta' \vdash_\omega \phi$ for some $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. By the induction hypothesis we can use **Cond** to obtain $\Gamma' \cup \{\psi \to \delta : \delta \in \Delta'\} \vdash_\omega \psi \to \phi$, and thus $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi$ by **W**.

**Cut:** $\Gamma \cup \Delta \vdash_\omega \Delta'$ and $\Gamma \cup \Delta \cup \Delta' \vdash_\omega \phi$, for some $\Delta'$. By the induction hypothesis on the first derivation (once for each $\delta' \in \Delta'$), $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \phi \to \delta'$ for each $\delta' \in \Delta'$. By the induction hypothesis on the second derivation, $\Gamma \cup \{\psi \to \delta : \delta \in \Delta \cup \Delta'\} \vdash_\omega \psi \to \phi$. By **Cut**, $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi$. ■

**Theorem 11 (Deduction Theorem)** The rule

$$\frac{\Gamma \cup \{\phi\} \vdash_\omega \psi}{\Gamma \vdash_\omega \phi \to \psi} \quad \mathbf{DT}$$

is admissible in $EC^\omega$. □

Proof If $\Gamma \cup \{\phi\} \vdash_\omega \psi$, then $\Gamma \cup \{\phi \to \phi\} \vdash_\omega \phi \to \psi$ by **Cond**. $\Gamma \vdash_\omega \phi \to \phi$ by **Ax** and **W**, and thus $\Gamma \vdash_\omega \phi \to \psi$ by **Cut**. ■

Now we are ready to show that consistent theories can be extended to maximal consistent theories. The proof relies on **DT**.

**Lemma 12 (Lindenbaum lemma for $EC^\omega$)** If $\Gamma$ is $EC^\omega$-consistent, then there exists an $\mathcal{L}_\triangledown$-maximal and $EC^\omega$-consistent $\Gamma'$ such that $\Gamma \subseteq \Gamma'$. □

Proof Recall **R***:

$$\bigcup_{j \in J} \{\alpha_j \to \neg K_k \psi : \psi \notin X_j\} \vdash_\omega \bigwedge_{j \in J} \alpha_j \to \triangledown_k X.$$

Formulae which can appear on the right of $\vdash_\omega$ in its instances will be said to have **R***-form. A special case of this schema is when $\bigwedge_j \alpha_j$ is a tautology (i.e., each $\alpha_j$ is), from which

$$\bigcup_{j \in J} \{\neg K_k \phi : \psi \notin X_j\} \vdash_\omega \triangledown_k X.$$

can be obtained. Now, $\Gamma' \supset \Gamma$ is constructed as follows. $\mathcal{L}_\triangledown$ is countable, so let $\phi_1, \phi_2, \ldots$ be an enumeration of $\mathcal{L}_\triangledown$ respecting the subformula relation (i.e., when $\phi_i$ is a subformula of $\phi_j$ then $i < j$).

$$\Gamma_0 = \Gamma$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\phi_{i+1}\} & \text{if } \Gamma_i \vdash_\omega \phi_{i+1} \\ \Gamma_i \cup \{\neg\phi_{i+1}\} & \text{if } \Gamma_i \nvdash_\omega \phi_{i+1} \text{ and } \phi_{i+1} \text{ does not have the } \mathbf{R*}\text{-form} \\ \Gamma_i \cup \{\neg\phi_{i+1}, K_k\psi\} & \text{if } \Gamma_i \nvdash_\omega \phi_{i+1} \text{ and } \phi_{i+1} \text{ has the } \mathbf{R*}\text{-form, where } \psi \text{ is} \\ & \text{arbitrary such that } \psi \notin X \text{ and } \Gamma_i \nvdash_\omega \neg K_k\psi \end{cases}$$

$$\Gamma' = \bigcup_{i=0}^{\omega} \Gamma_i$$

The existence of $\psi$ in the last clause in the definition of $\Gamma_{i+1}$ is verified as follows: since $\Gamma_i \nvdash_\omega \phi_{i+1}$, there must be, to prevent an application of **R***, at least one $\alpha_j$ and $\psi \notin X$ such that $\Gamma_i \nvdash_\omega \alpha_j \to \neg K_k\psi$. By construction (and the ordering of formulae), each $\alpha_j$ or its negation is included in $\Gamma_i$. If $\Gamma_i \vdash_\omega \neg\alpha_j$ then also $\Gamma_i \vdash_\omega \alpha_j \to \neg K_k\psi$, and this would be the case also if $\Gamma_i \vdash_\omega \neg K_k\psi$. So $\Gamma_i \vdash_\omega \alpha_j$ and $\Gamma_i \nvdash_\omega \neg K_k\psi$.

It is easy to see that $\Gamma'$ is maximal.

We show that each $\Gamma_i$ is consistent, by induction over $i$. For the base case, $\Gamma_0$ is consistent by assumption. For the inductive case, assume that $\Gamma_i$ is consistent. $\Gamma_{i+1}$ is constructed by one of the three cases in the definition:

1. $\Gamma_{i+1}$ is obviously consistent.
2. If $\Gamma_{i+1} = \Gamma_i \cup \{\neg\phi_{i+1}\} \vdash_\omega \bot$, then $\Gamma_i \vdash_\omega \phi_{i+1}$ by **DT** and **Prop**, contradicting the assumption in this case.
3. Consider first the special case (when all $\alpha_j$ are tautologies). Assume that $\Gamma_{i+1} = \Gamma_i \cup \{\neg \bigtriangledown_k X, K_k\psi\} \vdash_\omega \bot$. Then $\Gamma_i \vdash_\omega K_k\psi \to \bigtriangledown_k X$ by **DT** and **Prop** and by E4, since $\psi \notin X$, $\Gamma_i \vdash_\omega K_k\psi \to \neg \bigtriangledown_k X$, and thus $\Gamma_i \vdash_\omega \neg K_k\psi$ contradicting the assumption in this case.
   In the general case, assume that $\Gamma_{i+1} = \Gamma_i \cup \{\neg(\bigwedge_j \alpha_j \to \bigtriangledown_k X), K_k\psi\} \vdash_\omega \bot$:
   - i Then $\Gamma_i \vdash_\omega K_k\psi \to (\neg(\bigwedge_j \alpha_j \to \bigtriangledown_k X) \to \bot)$, i.e., $\Gamma_i \vdash_\omega K_k\psi \to (\bigwedge_j \alpha_j \to \bigtriangledown_k X)$, i.e., $\Gamma_i \vdash_\omega \bigwedge_j \alpha_j \to (K_k\psi \to \bigtriangledown_k X)$.
   - ii By assumption in the construction, $\Gamma_i \nvdash_\omega \neg(\bigwedge_j \alpha_j)$ (for otherwise it would prove $\bigwedge_j \alpha_j \to \bigtriangledown_k X$), but since $\bigwedge_j \alpha_j$ (as well as each $\alpha_j$) is a subformula of $\phi_{i+1}$, it or its negation is already included in $\Gamma_i$. But this means that $\Gamma_i \vdash_\omega \bigwedge_j \alpha_j$. Combined with (i), this gives $\Gamma_i \vdash_\omega K_k\psi \to \bigtriangledown_k X$, i.e., $\Gamma_i \vdash_\omega \neg K_k\psi \vee \bigtriangledown_k X$.
   - iii On the other hand, by E4, since $\psi \notin X : \Gamma_i \vdash_\omega \neg(K_k\psi \wedge \bigtriangledown_k X)$, i.e., $\Gamma_i \vdash_\omega \neg K_k\psi \vee \neg \bigtriangledown_k X$. Combined with (ii) this means that $\Gamma_i \vdash_\omega \neg K_k\psi$, but this contradicts the assumption in the construction of $\Gamma_{i+1}$.

Thus each $\Gamma_i$ is consistent.

To show that $\Gamma'$ is consistent, we first show that

$$\Gamma'' \vdash_\omega \phi \Rightarrow (\Gamma'' \subseteq \Gamma' \Rightarrow \phi \in \Gamma') \tag{1}$$

holds for all derivations $\Gamma'' \vdash_\omega \phi$, by induction over the derivation. The base cases are **Ax**, **MP** and **R***, and the inductive steps are **W** and **Cut**. Let $i$ be the index of the formula $\phi$, i.e. $\phi = \phi_i$.

**Ax:** If $\vdash_\omega \phi$, then $\phi \in \Gamma_i$ by the first case in the definition of $\Gamma_i$.

**MP:** $\Gamma'' = \{\phi', \phi' \to \phi\}$. If $\Gamma'' \subseteq \Gamma'$, there exists $k, l$ such that $\phi' \in \Gamma_k$ and $\phi' \to \phi \in \Gamma_l$. If $\phi \notin \Gamma'$, $\neg\phi \in \Gamma'$ by maximality, i.e. there exists a $m$ such that $\neg\phi \in \Gamma_m$. But then $\neg\phi, \phi', \phi' \to \phi \in \Gamma_{\max(k,l,m)}$, contradicting consistency of $\Gamma_{\max(k,l,m)}$.

**R*:** $\Gamma'' = \cup_{j \in J}\{\alpha_j \to \neg K_k\psi : \psi \notin X_j\}$ and $\phi = \bigwedge_j \alpha_j \to \bigtriangledown_k X$, where $X = \bigcap_j X_j$, and $\Gamma'' \subseteq \Gamma'$. If $\phi \notin \Gamma'$ then, by maximality, $\neg\phi \in \Gamma'$, and thus $\neg\phi \in \Gamma_i$. Then, by construction of $\Gamma_i$, $\Gamma_{i-1} \nvdash_\omega \phi$ (otherwise $\phi \in \Gamma'$) and

$K_k\psi \in \Gamma_i$ for some $\psi \notin X$. By the same argument as in point 3.(ii) above, $\Gamma_i \vdash_\omega \bigwedge_j \alpha_j$, and hence also $\Gamma' \vdash_\omega \bigwedge_j \alpha_j$. But then, for an appropriate $m$ (namely, for which $\phi_m = \alpha_j \to \neg K_k\psi$): $\Gamma_{m-1} \vdash_\omega \alpha_j$ and $\Gamma_{m-1} \vdash_\omega K_k\psi$, i.e., $\neg(\alpha_j \to \neg K_k\psi) \in \Gamma_m$, and so $\alpha_j \to \neg K_k\psi \notin \Gamma'$, which contradicts the assumption that $\Gamma'' \subseteq \Gamma'$.

**W:** $\Gamma'' = \Gamma''' \cup \Delta$, and $\Gamma''' \vdash_\omega \phi$. If $\Gamma'' \subseteq \Gamma'$, $\Gamma''' \subseteq \Gamma$ and by the induction hypothesis $\phi \in \Gamma'$.

**Cut:** $\Gamma'' \vdash_\omega \Delta$ and $\Gamma'' \cup \Delta \vdash_\omega \phi$. Let $\Gamma'' \subseteq \Gamma'$. By the induction hypothesis on the first derivation (once for each of the formulae in $\Delta$), $\Delta \subseteq \Gamma'$. Then $\Gamma'' \cup \Delta \subseteq \Gamma'$, and by the induction hypothesis on the second derivation $\phi \in \Gamma'$.

Thus (1) holds for all $\Gamma'' \vdash_\omega \phi$; particularly for $\Gamma' \vdash_\omega \phi$. Consistency of $\Gamma'$ follows: if $\Gamma' \vdash_\omega \bot$, then $\bot \in \Gamma'$, i.e. $\bot \in \Gamma_l$ for some $l$, contradicting the fact that each $\Gamma_l$ is consistent. ∎

The following Lemma is needed in the proof of the thereafter following Lemma stating satisfiability of maximal consistent theories.

**Lemma 13** Let $\Gamma' \subseteq \mathcal{L}_\triangledown$ be an $\mathcal{L}_\triangledown$-maximal and $EC^\omega$-consistent theory. If there exists an $X'$ such that $\Gamma' \vdash_\omega \triangledown_i X'$, then there exists an $X$ such that $\Gamma' \vdash_\omega \Diamond_i X$. □

PROOF Let $\Gamma'$ be maximal consistent, and let $\Gamma' \vdash_\omega \triangledown_i X'$. Let

$$X = \bigcap_{Y \subseteq X' \text{ and } \Gamma' \vdash_\omega \triangledown_i Y} Y$$

Since every $Y$ is included in the finite set $X'$, $X$ is finite, and $\Gamma' \vdash_\omega \triangledown_i X$ can be obtained by a finite number of applications of E3. Let

$$Z = \bigcup_{\Gamma' \vdash_\omega \triangle_i Y} Y$$

If $\Gamma' \vdash_\omega \triangle_i Y$, then $Y \subseteq X$, since otherwise $\Gamma'$ would be inconsistent by E4. Thus $Z$ is finite. By a finite number of applications of E2, $\Gamma' \vdash_\omega \triangle_i Z$. If $Z \not\subseteq X$, then $\Gamma'$ would be inconsistent by E4, so $Z \subseteq X$. We now show that $X \subseteq Z$. Assume the opposite: $\phi \in X$ but $\phi \notin Z$ for some $\phi$. Let $X^- = X \setminus \{\phi\}$. $\Gamma' \not\vdash_\omega K_i\phi$, since otherwise $\phi \in Z$ by definition of $Z$. By maximality, $\Gamma' \vdash_\omega \neg K_i\phi$. By E5, $\Gamma' \vdash_\omega \triangledown_i X^-$ – but by construction of $X$ it follows that $X \subseteq X^-$ which is a contradiction. Thus, $X = Z$, and $\Gamma' \vdash_\omega \Diamond_i X$. ∎

**Lemma 14** Every maximal $EC^\omega$-consistent $\mathcal{L}_\triangledown$ theory is satisfiable. □

PROOF Let $\Gamma$ be maximal and consistent. We construct the following SSA, which is intended to satisfy $\Gamma$:

$$M^\Gamma = (\{s\}, \sigma^\Gamma)$$
$$\sigma^\Gamma(s)(p) = \mathbf{true} \Leftrightarrow \Gamma \vdash_\omega p \text{ when } p \in \Theta$$
$$\sigma^\Gamma(s)(K_i\phi) = \mathbf{true} \Leftrightarrow \phi \in X_i^\Gamma$$

where:

$$X_i^\Gamma = \begin{cases} Z \text{ where } \Gamma \vdash_\omega \Diamond_i Z \text{ if there is an } X' \text{ such that } \Gamma \vdash_\omega \triangledown_i X' \\ \{\gamma : \Gamma \vdash_\omega K_i\gamma\} \qquad \text{otherwise} \end{cases}$$

In the definition of $X_i^\Gamma$, the existence of a $Z$ such that that $\Gamma \vdash_\omega \Diamond_i Z$ in the case that there exists an $X'$ such that $\Gamma \vdash_\omega \triangledown_i X'$ is guaranteed by Lemma 13. We show, by structural induction over $\phi$, that

$$(M^\Gamma, s) \models \phi \Longleftrightarrow \Gamma \vdash_\omega \phi \tag{2}$$

This is a stronger statement than the lemma; the lemma is given by the direction to the left. We use three base cases: when $\phi$ is in $\Theta$, $\phi = K_i\psi$ and $\phi = \triangledown_i X$. The first base case and the two inductive steps negation and conjunction are trivial, so we show only the two interesting base cases. For each base case we consider the situations when $X_i^\Gamma$ is given by a) the first and b) the second case in its definition.

- $\phi = K_i\psi$: $(M^\Gamma, s) \models K_i\psi$ iff $\psi \in X_i^\Gamma$.
  - ⇒) Let $\psi \in X_i^\Gamma$. In case a), $X_i^\Gamma = Z$ where $\Gamma \vdash_\omega \Diamond_i Z$ and by **KS**, $\Gamma \vdash_\omega K_i\psi$. In case b), $\Gamma \vdash_\omega K_i\psi$ by construction of $X_i^\Gamma$.
  - ⇐) Let $\Gamma \vdash_\omega K_i\psi$. In case a), $\Gamma \vdash_\omega \triangledown_i Z$ and thus $\psi \in Z = X_i^\Gamma$ by E4 and consistency of $\Gamma$. In case b), $\psi \in X_i^\Gamma$ by construction.
- $\phi = \triangledown_i X$: $(M^\Gamma, s) \models \triangledown_i X$ iff $X_i^\Gamma \subseteq X$.
  - ⇒) Let $X_i^\Gamma \subseteq X$. In case a), $\Gamma \vdash_\omega \Diamond_i Z$ where $Z = X_i^\Gamma \subseteq X$, so $\Gamma \vdash_\omega \triangledown_i X$ by **KG**. In case b), $X_i^\Gamma$ must be finite, since $X$ is finite. For any $\psi \notin X_i^\Gamma$, $\Gamma \not\vdash_\omega K_i\psi$ by construction of $X_i^\Gamma$, and $\Gamma \vdash_\omega \neg K_i\psi$ by maximality. Thus, by **R*** (with $J = \{1\}$, $\alpha_1 = \top$ and $X_1 = X_i^\Gamma$), $\Gamma \vdash_\omega \triangledown_i X_i^\Gamma$, contradicting the assumption in case b). Thus, case b) is impossible.
  - ⇐) Let $\Gamma \vdash_\omega \triangledown_i X$. In case a), $\Gamma \vdash_\omega \triangle_i Z$ and by E4 and consistency of $\Gamma$ $X_i^\Gamma = Z \subseteq X$. Case b) is impossible by definition. ■

**Theorem 15** $EC^\omega$ is a sound and strongly complete axiomatization of standard syntactic assignments, in the language $\mathcal{L}_\triangledown$. □

Proof Soundness follows from Lemma 6, and the easily seen facts that **MP** and **R*** are logical consequences and that **W** and **Cut** preserve logical consequence, by induction over the definition of the derivation relation. Strong completeness follows from Lemmas 12 and 14. ■

### 4.2 A System for a Weaker Language

In the previous section we proved strong completeness of $EC^\omega$ by using **R***. It turns out that strong completeness can be proved without **R***, if we restrict the logical language slightly. The restriction is that for some arbitrary primitive proposition $\hat{p} \in \Theta$, $K_i\hat{p}$ and $\triangledown_i X$ are not well-formed formulae for any $i$ and any $X$ with $\hat{p} \in X$. The semantics is not changed; we are still interpreting the language in SSAs over $\mathcal{L}(\Theta, n)$ as described in Sections 2 and 3. Thus, in the restricted logic agents can know something which is not expressible in the logical language.

$\mathcal{L}_{\triangledown}^{\hat{p}} \subset \mathcal{L}_{\triangledown}$ is the restricted language for a given primitive proposition $\hat{p}$.

**Definition 16 ($\mathcal{L}_{\triangledown}^{\hat{p}}$)** Given a set of primitive propositions $\Theta$, a proposition $\hat{p} \in \Theta$ and a number of agents $n$, $\mathcal{L}_{\triangledown}^{\hat{p}}(\Theta, n)$ (or just $\mathcal{L}_{\triangledown}^{\hat{p}}$) is the least set such that:

- $\Theta \subseteq \mathcal{L}_{\triangledown}^{\hat{p}}$
- If $\phi, \psi \in \mathcal{L}_{\triangledown}^{\hat{p}}$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{L}_{\triangledown}^{\hat{p}}$
- If $\phi \in (\mathcal{L} \setminus \{\hat{p}\})$ and $i \in \Sigma$ then $K_i\phi \in \mathcal{L}_{\triangledown}^{\hat{p}}$
- If $X \in \wp^{fin}(\mathcal{L} \setminus \hat{p})$ and $i \in \Sigma$ then $\triangledown_i X \in \mathcal{L}_{\triangledown}^{\hat{p}}$ □

The finitary logical system $EC^{\hat{p}}$ is defined by the same axiom schemas as $EC^{\omega}$. The two systems do not, however, have the same axioms since they are defined for different languages – the extensions of the schemas are different. The derivation relation for $EC^{\hat{p}}$ is defined by the axioms and the derivation rule modus ponens. Particularly, the infinitary derivation clause **R*** from $EC^{\omega}$ is not included.

**Definition 17 ($EC^{\hat{p}}$)** $EC^{\hat{p}}$ is the logical system for the language $\mathcal{L}_{\triangledown}^{\hat{p}}$ consisting of the following axiom schemata:

| | | |
|---|---|---|
| All substitution instances of tautologies of propositional calculus | | **Prop** |
| $(\triangledown_i X \wedge \triangledown_i Y) \rightarrow \triangledown_i (X \cap Y)$ | | E3 |
| $\neg(\triangle_i X \wedge \triangledown_i Y)$ | when $X \not\subseteq Y$ | E4 |
| $(\triangledown_i (Y \cup \{\gamma\}) \wedge \neg K_i\gamma) \rightarrow \triangledown_i Y$ | | E5 |
| $\triangledown_i X \rightarrow \triangledown_i Y$ | when $X \subseteq Y$ | **KG** |

The derivation relation $\vdash_{EC^{\hat{p}}}$ – written $\vdash_{\hat{p}}$ for simplicity – between sets of $\mathcal{L}_{\triangledown}^{\hat{p}}$ formulae and single $\mathcal{L}_{\triangledown}^{\hat{p}}$ formulae is the smallest relation closed under the following conditions:

| | | |
|---|---|---|
| $\Gamma \vdash_{\hat{p}} \phi$ | when $\phi \in \Gamma$ | **Prem** |
| $\Gamma \vdash_{\hat{p}} \phi$ | when $\phi$ is an axiom | **Ax** |
| $\dfrac{\Gamma \vdash_{\hat{p}} \phi, \Gamma \vdash_{\hat{p}} \phi \rightarrow \psi}{\Gamma \vdash_{\hat{p}} \psi}$ | | **MP** |

□

It is easy to see that E1, E2, **KS** and **DT** are derivable in $EC^{\omega}$.

The restriction $\mathcal{L}_{\triangledown}^{\hat{p}} \subset \mathcal{L}_{\triangledown}$ is sufficient to prove strong completeness without **R*** in a manner very similar to the proof in Section 4.1. The first step, existence of maximal consistent extensions, can now be proved by the standard proof since the system is finitary.

**Lemma 18 (Lindenbaum lemma for $EC^{\hat{p}}$)** If $\Gamma$ is $EC^{\hat{p}}$-consistent, then there exists an $\mathcal{L}_{\triangledown}^{\hat{p}}$-maximal and $EC^{\hat{p}}$-consistent $\Gamma'$ such that $\Gamma \subseteq \Gamma'$. □

Second, we establish the result corresponding to Lemma 13 for $\mathcal{L}_{\triangledown}^{\hat{p}}$ and $EC^{\hat{p}}$.

**Lemma 19** Let $\Gamma' \subseteq \mathcal{L}_{\triangledown}^{\hat{p}}$ be a $\mathcal{L}_{\triangledown}^{\hat{p}}$-maximal and $EC^{\hat{p}}$-consistent theory. If there exists a $X'$ such that $\Gamma' \vdash_{\hat{p}} \triangledown_i X'$, then there exists a $X$ such that $\Gamma' \vdash_{\hat{p}} \diamondsuit_i X$. □

Proof The proof is essentially the same as for Lemma 13, for the language $\mathcal{L}_{\triangledown}^{\hat{p}}$ instead of $\mathcal{L}_{\triangledown}$ (note that in that proof we did not rely on **R***, and that $\hat{p} \notin X$ since $X \subseteq X'$). ∎

Third, we show satisfiability.

**Lemma 20** Every maximal $EC^{\hat{p}}$-consistent $\mathcal{L}_{\triangledown}^{\hat{p}}$ theory is satisfiable. □

Proof Let $\Gamma$ be maximal and consistent. The proof is very similar to that of the corresponding result for $EC^{\omega}$ (Lemma 14). We construct the following SSA, which is intended to satisfy $\Gamma$:

$$M^{\Gamma} = (\{s\}, \sigma^{\Gamma})$$
$$\sigma^{\Gamma}(s)(p) = \mathbf{true} \Leftrightarrow \Gamma \vdash_{\hat{p}} p \text{ when } p \in \Theta$$
$$\sigma^{\Gamma}(s)(K_i\phi) = \mathbf{true} \Leftrightarrow \phi \in X_i^{\Gamma}$$

where:

$$X_i^{\Gamma} = \begin{cases} Z \text{ where } \Gamma \vdash_{\hat{p}} \diamondsuit_i Z & \text{if there is an } X' \text{ such that } \Gamma \vdash_{\hat{p}} \triangledown_i X' \\ \{\gamma : \Gamma \vdash_{\hat{p}} K_i\gamma\} \cup \{\hat{p}\} & \text{if } \forall_{X'} \Gamma \nvdash_{\hat{p}} \triangledown_i X' \text{ and } \bigcup_{\Gamma \vdash_{\hat{p}} \triangle_i Y} Y \text{ is finite} \\ \{\gamma : \Gamma \vdash_{\hat{p}} K_i\gamma\} & \text{if } \forall_{X'} \Gamma \nvdash_{\hat{p}} \triangledown_i X' \text{ and } \bigcup_{\Gamma \vdash_{\hat{p}} \triangle_i Y} Y \text{ is infinite} \end{cases}$$

The existence of $Z$ is guaranteed by Lemma 19, and, again, we show, by structural induction over $\phi$, that

$$(M^{\Gamma}, s) \models \phi \Longleftrightarrow \Gamma \vdash_{\hat{p}} \phi \quad (3)$$

for all $\phi \in \mathcal{L}_{\triangledown}^{\hat{p}}$. As in the proof of Lemma 14 we only show the epistemic base cases. For each base case we consider the situations when

**a)** there is an $X'$ such that $\Gamma \vdash_{\hat{p}} \triangledown_i X'$ or
**b)** $\Gamma \nvdash_{\hat{p}} \triangledown_i X'$ for every $X'$

corresponding to the first and to the second and third cases in the definition of $X_i^{\Gamma}$, respectively.

- $\phi = K_i\psi$: $(M^{\Gamma}, s) \models K_i\psi$ iff $\psi \in X_i^{\Gamma}$.
  ⇒) Let $\psi \in X_i^{\Gamma}$. In case a), $X_i^{\Gamma} = Z$ where $\Gamma \vdash_{\hat{p}} \diamondsuit_i Z$ and by **KS**, $\Gamma \vdash_{\hat{p}} K_i\psi$. In case b), $\psi \neq \hat{p}$ (since $K_i\psi \in \mathcal{L}_{\triangledown}^{\hat{p}}$) and thus $\Gamma \vdash_{\hat{p}} K_i\psi$ by construction of $X_i^{\Gamma}$.

$\Leftarrow$) Let $\Gamma \vdash_{\hat{p}} K_i\psi$. In case a), $\Gamma \vdash_{\hat{p}} \triangledown_i Z$ and thus $\psi \in Z = X_i^\Gamma$ by E4 and consistency of $\Gamma$. In case b), $\psi \in X_i^\Gamma$ by construction.

– $\phi = \triangledown_i X$: $(M^\Gamma, s) \models \triangledown_i X$ iff $X_i^\Gamma \subseteq X$.

$\Rightarrow$) Let $X_i^\Gamma \subseteq X$. In case a), $\Gamma \vdash_{\hat{p}} \Diamond_i Z$ where $Z = X_i^\Gamma \subseteq X$, so $\Gamma \vdash_{\hat{p}} \triangledown_i X$ by **KG**. In case b), if $\hat{p} \in X_i^\Gamma$ then $\hat{p} \in X$ which is impossible since $\triangledown_i X$ is a formula. But if $\hat{p} \notin X_i^\Gamma$ then $X_i^\Gamma$ is infinite (by construction) which is also impossible since $X$ is finite – thus case b) is impossible.

$\Leftarrow$) Let $\Gamma \vdash_{\hat{p}} \triangledown_i X$. In case a), $\Gamma \vdash_{\hat{p}} \triangle_i Z$ and by E4 and consistency of $\Gamma$ $X_i^\Gamma = Z \subseteq X$. Case b) is impossible by definition. ■

**Theorem 21** $EC^{\hat{p}}$ is a sound and strongly complete axiomatization of standard syntactic assignments, in the language $\mathcal{L}_\triangledown^{\hat{p}}$. □

PROOF Soundness follows from the soundness of $EC^\omega$ and the fact that $\Gamma \vdash_{\hat{p}} \phi$ implies $\Gamma \vdash_\omega \phi$, the latter which can be seen by induction on the length of a proof in $EC^{\hat{p}}$ (every $\mathcal{L}_\triangledown^{\hat{p}}$ formula is also a $\mathcal{L}_\triangledown$ formula): the base case **Prem** follows by R1 (Lemma 9), the base case **Ax** follows by **Ax** and **W**, and the inductive case **MP** follows by **MP**, **W** and **Cut**. Strong completeness follows from Lemmas 20 and 18. ■

## 5 Discussion

We introduced a "knows at most" operator in order to increase the expressiveness of the epistemic language with respect to standard syntactic assignments, and investigated strong axiomatization of the resulting logic. The new operator destroyed semantic compactness and thus the possibility of a strongly complete finitary axiomatization, but we presented a strongly complete infinitary axiomatization. An interesting result is that we have a strongly complete finitary axiomatization if we make the assumption that the agents can know something which is not expressible in the logical language. The results are a contribution to the logical foundation of multi-agent systems.

Related work include classical syntactic treatment of knowledge [6–8, 1], modelled in a possible worlds framework by [1] as described in Section 2. The $\triangledown_i$ operator is new in the context of syntactic models. It is, however, similar to Levesque's *only knowing* operator **O** [9]. **O**$\alpha$ means that the agent does not know more than $\alpha$, but knowledge in this context means knowledge closed under logical consequence and "only knowing $\alpha$" is thus different from "knowing at most" a general set of formulae. The relation between these concepts is an interesting possibility for future work.

In [4] we investigate the special case of agents who can know only *finitely* many syntactic formulae at the same time. Completeness results for such finitely restricted agents build upon the results presented in this paper. Another possibility for future work is to study other special classes of SSAs.

In this paper we have only studied the *static* aspect of syntactic knowledge. In [10], we discuss how syntactic knowledge can evolve as a result of reasoning and communication, i.e. a *dynamic* aspect of knowledge.

## References


1. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning About Knowledge. The MIT Press, Cambridge, Massachusetts (1995)
2. Meyer, J.J.C., van der Hoek, W.: Epistemic Logic for AI and Computer Science. Cambridge University Press, Cambridge, England (1995)
3. Ågotnes, T., Walicki, M.: A logic for reasoning about agents with finite explicit knowledge. In Tessem, B., Ala-Siuru, P., Doherty, P., Mayoh, B., eds.: Proc. of the 8th Scandinavian Conference on Artificial Intelligence. Frontiers in Artificial Intelligence and Applications, IOS Press (2003) 163–174
4. Ågotnes, T., Walicki, M.: Complete axiomatizations of finite syntactic epistemic states. In: To appear in the Proceedings of the 3rd International Workshop on Declarative Agent Languages and Technologies (DALT 2005), Utrecht, the Netherlands (2005)
5. de Lavalette, G.R., Kooi, B., Verbrugge, R.: Strong completeness for propositional dynamic logic. In Balbiani, P., Suzuki, N.Y., Wolter, F., eds.: Preliminary Proceedings of AiML2002, Institut de Recherche en Informatique de Toulouse IRIT (2002) 377–393
6. Eberle, R.A.: A logic of believing, knowing and inferring. Synthese **26** (1974) 356–382
7. Moore, R.C., Hendrix, G.: Computational models of beliefs and the semantics of belief sentences. Technical Note 187, SRI International, Menlo Park, CA (1979)
8. Halpern, J.Y., Moses, Y.: Knowledge and common knowledge in a distributed environment. Journal of the ACM **37** (1990) 549–587
9. Levesque, H.J.: All I know: a study in autoepistemic logic. Artificial Intelligence **42** (1990) 263–309
10. Ågotnes, T., Walicki, M.: Syntactic knowledge: A logic of reasoning, communication and cooperation. In: Proceedings of the Second European Workshop on Multi-Agent Systems (EUMAS), Barcelona, Spain (2004)